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1987 J. Phys. A: Math. Gen. 20 L21

(http://iopscience.iop.org/0305-4470/20/1/005)

J. Phys. A: Math. Gen. 20 (1987) L21–L23. Printed in the UK

LETTER TO THE EDITOR

Absence of localisation in the almost Mathieu equation

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Received 29 September 1986

Abstract. We consider the discrete Schrödinger operator on $\mathbb{Z}$ with the potential $\lambda \cos 2\pi(\alpha n+\theta)$. This one-dimensional model occurs in the study of an electron in a two-dimensional periodic potential with an uniform magnetic field. First we prove that for every α and for $\lambda<2$ this operator has no eigenvalue (i.e. localised state). Furthermore at $\lambda=2$, the eigenvalues (if they exist) belong to the set where the Lyapunov exponent vanishes and the associated eigenvectors are in $l^2(\mathbb{Z})$ but are not summable.

In this letter we consider the following family of operators $H_{\lambda,\theta}$ acting on $l^2(\mathbb{Z})$:

$$(H_{\lambda,\theta}u)(n)=u(n+1)+u(n-1)+\lambda \cos 2\pi(\alpha n+\theta)u(n) \tag{1}$$

where u is a sequence in $l^2(\mathbb{Z})$ (i.e. such that $\Sigma_n|u(n)|^2<+\infty$) and α is an irrational number. This equation has been used to describe a two-dimensional electron in a periodic potential with a uniform perpendicular magnetic field; then (1) occurs in the two limits where the magnetic field (respectively the periodic potential) is weak in which case α (respectively $1/\alpha$) is the number of flux quanta per unit cell. Since the early works [1, 2] many exact results about the spectral resolution of $H_{\lambda,\theta}$ have been obtained. For small λ, by a perturbation technique it is proved [3] that there exists a component of an absolutely continuous spectrum provided α is a good irrational (Diophantine) number. This component corresponds to quasiperiodic waves obtained by perturbation of the plane waves at $\lambda=0$. This set of quasiperiodic waves spans the entire Hilbert space only in the limit $\lambda=0$; but one expects (and numerical studies suggest) that this is an artefact of the perturbation technique and that the whole spectrum is absolutely continuous as long as $\lambda<2$ and α is a good irrational number. In this letter we prove a result which concurs with this conjecture; namely, we obtain the absence of eigenvalues (localised states) for any α and any λ strictly smaller than 2. This proof is a rigorous version of a duality result contained in [1]. Absence of eigenvalues has already been obtained when α is a Liouville number [4] and for almost every α in the case of the one-dimensional quasicrystals [5] where the cosine is replaced by a piecewise constant function. On the other hand, for $\lambda>2$ an idea of Pastur [6] shows that there does not exist an absolutely continuous spectrum since the Lyapunov exponent $\gamma(E)$ of the product of Jacobi matrices associated with the eigenvalue problem $H_{\lambda,\theta}u=Eu$ is always strictly positive [2, 7, 8]. Furthermore it has been proved [9] that for large λ and good irrational numbers there exist eigenvalues with exponentially decaying eigenvectors. The case $\lambda=2$ is of particular interest; it corresponds to the so-called Harper's equation and describes a two-dimensional crystal with square symmetry. In this case one expects that the spectrum (as a set) is of zero Lebesgue

0305-4470/87/010021+03\$02.50

measure and that the spectral measure is purely singular continuous. In this case, we prove that the eigenvalues (if they exist) do belong to the set where the Lyapunov exponent vanishes and that the corresponding eigenvectors are in $l^2(\mathbb{Z})$ (by definition) but are not in $l^1(\mathbb{Z})$. Finally we mention that there are many other interesting results about this model; references can be found in [10, 11].

Let us assume that there exists an eigenvalue E of $H_{\lambda,\theta}$ for some $\lambda < 2$ and some θ. Let us denote by u the corresponding normalised eigenvector and $\hat{u}$ its Fourier transform; that is

$$\hat{u}(k) = \sum_n u(n) \exp(\mathrm{i}2\pi kn). \tag{2}$$

Then it is well known, by the Aubry-André duality [1], that the sequence $v(n) = \hat{u}(\tau + \alpha n) \exp(\mathrm{i}n\theta)$, for fixed τ, is formally a solution of

$$H_{4/\lambda,\tau} v = 2E/\lambda v \tag{3}$$

as can be checked by direct computation. Now let us specify the sense in which this solution actually exists. The function $\hat{u}(k)$ is the Fourier transform of a sequence in $l^2(\mathbb{Z})$ and thus is defined as an L^2 function of the unit circle with the normalised Lebesque measure; furthermore its L^2 norm is one. This function is only defined up to a set of zero Lebesque measure so that (3) is satisfied for almost every τ. Since $\hat{u}(k)$ has norm one in $L^2(0, 1)$ we have:

$$\int \mathrm{d}\tau \sum_{n\in\mathbb{Z}} |\hat{u}(\alpha n + \tau)|^2/|n|^{1+\varepsilon} = \sum_{n\in\mathbb{Z}} 1/|n|^{1+\varepsilon} < +\infty \tag{4}$$

for any $\varepsilon > 0$. Consequently for any $\varepsilon > 0$, and for almost every τ, there exists a constant $C_\varepsilon(\tau)$ such that

$$|\hat{u}(\alpha n + \tau)| < C_\varepsilon(\tau)|n|^{1/2+\varepsilon/2} \tag{5}$$

for all $n \in \mathbb{Z}$ where $\hat{u}(\alpha n + \tau) \exp(\mathrm{i}n\theta)$ is a solution of (3). Furthermore $\hat{u}(k)$ cannot vanish almost everywhere thus $v(n)$ cannot be identically zero for almost every τ and there is a non-trivial solution of (3) satisfying (5) for a set of τ with positive Lebesgue measure. This in turn implies that the Lyapunov exponent of the product of Jacobi matrices associated with (3) is zero as shown below.

Let us recall that $H_{\lambda,\theta} u = Eu$ can be rewritten as

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} E - \lambda \cos 2\pi(\alpha n + \theta) & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = M_n \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}.$$

For almost every θ

$$\lim_{n\to\infty} \frac{1}{|n|} \log \left\| \prod_{i=1}^{n} M_i \right\|$$

exists and is almost certainly constant (independent of θ). This limit is the Lyapunov exponent $\gamma(E)$ and Oseledec's theorem ensures that for almost every θ, any solution of $H_{\lambda,\theta} u = Eu$ has to increase exponentially at $+\infty$ or $-\infty$ with the rate given by $\gamma(E)$. Thus inequality (5) (for a set of τ of positive Lebesgue measure) does imply that the Lyapunov exponent associated with the eigenvalue equation (3) is zero. On the other hand, it is known that for $\lambda > 2$, and for any energy, $\gamma(E)$ is strictly positive. Consequently there cannot exist any eigenvalue E of $H_{\lambda,\theta}$ for any $\lambda < 2$ and for any θ: otherwise γ should vanish at $\lambda' = 4/\lambda > 2$ and $E' = 2E/\lambda$.

If $\lambda = 2$, since λ and E are not modified by the Fourier transform of (1), the previous results yield that eigenvalues can only occur on the set where the Lyapunov exponent vanishes. Furthermore, if an eigenfunction u belongs to $l^1(\mathbb{Z})$ then its Fourier transform $\hat{u}(k)$ is continuous. Thus, for any τ, (3) has a quasiperiodic solution. This solution cannot be zero since by continuity $\hat{u}$ should vanish all over the circle (as soon as α is irrational). In particular for $\tau = \theta$ this quasiperiodic solution $\hat{u}(\alpha n + \theta)\exp(\mathrm{i}n\theta)$ exists and should have a constant Wronskian with the initial eigenfunction u which is impossible. Consequently, any eigenfunction does belong to $l^2(\mathbb{Z})\backslash l^1(\mathbb{Z})$.

It may be noticed that at $\lambda = 2$, one expects the Lyapunov exponent to vanish, at the most, on a set of zero Lebesque measure and moreover the spectrum to be of zero Lebesgue measure. In this case the spectral measure has to be singular and the difference between non-$l^1(\mathbb{Z})$ eigenvectors and a singular continuous spectrum may be difficult to highlight.

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